

ORBITS OF SMOOTH FUNCTIONS ON 2-TORUS AND THEIR HOMOTOPY TYPES

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ABSTRACT. Let $f : T^2 \rightarrow \mathbb{R}$ be a Morse function on 2-torus T^2 such that its Kronrod-Reeb graph $\Gamma(f)$ has exactly one cycle, i.e. it is homotopy equivalent to S^1 . Under some additional conditions we describe a homotopy type of the orbit of f with respect to the action of the group of diffeomorphism of T^2 .

This result holds for a larger class of smooth functions $f : T^2 \rightarrow \mathbb{R}$ having the following property: for every critical point z of f the germ of f at z is smoothly equivalent to a homogeneous polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors.

1. INTRODUCTION

Let M be a smooth oriented surface. For a closed (possibly empty) subset $X \subset M$ denote by $\mathcal{D}(M, X)$ the group of diffeomorphisms of M fixed on X . This group naturally acts from the right on the space of smooth functions $C^\infty(M)$ by following rule: if $h \in \mathcal{D}(M, X)$ and $f \in C^\infty(M)$ then the result of the action of h on f is the composition map

$$f \circ h : M \xrightarrow{h} M \xrightarrow{f} \mathbb{R}. \quad (1.1)$$

For $f \in C^\infty(M)$ let

$$\mathcal{S}(f, X) = \{f \in \mathcal{D}(M, X) \mid f \circ h = f\}, \quad \mathcal{O}(f, X) = \{f \circ h \mid h \in \mathcal{D}(M, X)\}.$$

be respectively the *stabilizer* and the *orbit* of f under the action (1.1).

Endow on $\mathcal{D}(M, X)$, $C^\infty(M)$ and their subspaces $\mathcal{S}(f, X)$ and $\mathcal{O}(f, X)$ with the corresponding Whitney C^∞ -topologies. Let also $\mathcal{S}_{\text{id}}(f, X)$ be the path component of the identity map id_M in $\mathcal{S}(f, X)$, $\mathcal{D}_{\text{id}}(M, X)$ be the path component of id_M in $\mathcal{D}(M, X)$, and $\mathcal{O}_f(f, X)$ be the path component of f in $\mathcal{O}(f, X)$. If $X = \emptyset$ then we omit it from notation and write $\mathcal{D}(M) = \mathcal{D}(M, \emptyset)$, $\mathcal{S}(f) = \mathcal{S}(f, \emptyset)$, $\mathcal{O}(f) = \mathcal{O}(f, \emptyset)$, and so on.

We will assume that all the homotopy groups of $\mathcal{O}(f, X)$ will have f as a base point, and all homotopy groups of the groups of diffeomorphisms and the corresponding stabilizers of f are based at id_M . For instance $\pi_k(\mathcal{O}(f, X))$ will always mean $\pi_k(\mathcal{O}(f, X), f)$. Notice that the latter group is also isomorphic with $\pi_k(\mathcal{O}_f(f, X), f)$.

Since $\mathcal{D}(M, X)$ and $\mathcal{S}(f, X)$ are topological groups, it follows that the homotopy sets $\pi_0 \mathcal{D}(M, X)$, $\pi_0 \mathcal{S}(f, X)$, and $\pi_1(\mathcal{D}(M, X), \mathcal{S}(f, X))$ have natural groups structures such that

$$\pi_0 \mathcal{D}(M, X) \cong \mathcal{D}(M, X) / \mathcal{D}_{\text{id}}(M, X), \quad \pi_0 \mathcal{S}(f, X) \cong \mathcal{S}(f, X) / \mathcal{S}_{\text{id}}(f, X),$$

and in the following part of exact sequence of homotopy groups of the pair $(\mathcal{D}(M, X), \mathcal{S}(f, X))$

$$\cdots \rightarrow \pi_1 \mathcal{D}(M, X) \xrightarrow{q} \pi_1(\mathcal{D}(M, X), \mathcal{S}(f, X)) \xrightarrow{\partial} \pi_0 \mathcal{S}(f, X) \xrightarrow{i} \pi_0 \mathcal{D}(M, X) \quad (1.2)$$

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all maps are homomorphisms.

Moreover, $q(\pi_1 \mathcal{D}(M, X))$ is contained in the center of $\pi_1(\mathcal{D}(M, X), \mathcal{S}(f, X))$.

Recall that two smooth germs $f, g : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ are said to be *smoothly equivalent* if there exist germs of diffeomorphisms $h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $\phi \circ g = f \circ h$.

Definition 1.1. Denote by $\mathcal{F}(M)$ a subset in $C^\infty(M)$ which consists of functions f having the following two properties:

- f takes a constant value at each connected components of ∂M , and all critical points of f are contained in the interior of M ;
- for each critical point z of f the germ of f at z is smoothly equivalent to a **homogeneous polynomial** $f_z : \mathbb{R}^2 \rightarrow \mathbb{R}$ **without multiple factors**.

Suppose a smooth germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ has a critical point $0 \in \mathbb{R}^2$. This point is called *non-degenerate* if f is smoothly equivalent to a homogeneous polynomial of the form $\pm x^2 \pm y^2$.

Denote by $\text{Morse}(M)$ the subset of $C^\infty(M)$ consisting of Morse functions, that is functions having only *non-degenerate* critical points. It is well known that $\text{Morse}(M)$ is open and everywhere dense in $C^\infty(M)$. Since $\pm x^2 \pm y^2$ has no multiple factors, we get the following inclusion $\text{Morse}(M) \subset \mathcal{F}(M)$.

Remark 1.2. A homogeneous polynomial $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has critical points only if $\deg f \geq 2$, and in this case the origin is always a critical point of f . If f has no multiple factors, then the origin 0 is a unique critical point. Moreover, 0 is non-degenerate $\deg f = 2$, and degenerate for $\deg f \geq 3$, see [10, §7].

Now let $f \in \mathcal{F}(M)$ and $c \in \mathbb{R}$. A connected component C of the level set $f^{-1}(c)$ is said to be *critical* if C contains at least one critical point of f . Otherwise C is called *regular*. Consider a partition of M into connected component of level sets of f . It is well known that the corresponding factor-space $\Gamma(f)$ has a structure of a finite one-dimensional CW-complex and is called *Kronrod-Reeb graph* or simply *KR-graph* of the function f . In particular, the vertices of $\Gamma(f)$ are critical components of level sets of f .

It is usually said that this graph was introduced by G. Reeb in [16], however it was used before by A. S. Kronrod in [5] for studying functions on surfaces. Applications of $\Gamma(f)$ to study Morse functions on surfaces are given e.g. in [1, 7, 6, 17, 18, 15].

In a series of papers the first author calculated homotopy types of spaces $\mathcal{S}(f)$ and $\mathcal{O}(f)$ for all $f \in \mathcal{F}(M)$. These results are summarized in Theorem 1.3 below.

Denote also

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M), \quad \mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M, X).$$

Thus $\mathcal{S}'(f, X)$ consists of diffeomorphisms h preserving f , fixed on X and isotopic to id_M relatively X , though the isotopy between h and id_M is not required to be f -preserving.

Theorem 1.3. [9, 11, 12]. Let $f \in \mathcal{F}(M)$ and X be a finite (possibly empty) union of regular components of certain level sets of function f . Then the following statements hold true.

- (1) $\mathcal{O}_f(f, X) = \mathcal{O}_f(f, X \cup \partial M)$, and so

$$\pi_k \mathcal{O}(f, X) \cong \pi_k \mathcal{O}(f, X \cup \partial M), \quad k \geq 1.$$

(2) The following map

$$p : \mathcal{D}(M, X) \longrightarrow \mathcal{O}(f, X), \quad p(h) = f \circ h.$$

is a Serre fibration with fiber $\mathcal{S}(f, X)$, i.e. it has homotopy lifting property for CW-complexes. This implies that

- (a) $p(\mathcal{D}_{\text{id}}(M, X)) = \mathcal{O}_f(f, X)$;
- (b) the restriction map

$$p|_{\mathcal{D}_{\text{id}}(M, X)} : \mathcal{D}_{\text{id}}(M, X) \longrightarrow \mathcal{O}_f(f, X) \quad (1.3)$$

is also a Serre fibration with fiber $\mathcal{S}'(f, X)$;

- (c) for each $k \geq 0$ we have an isomorphism $j_k : \pi_k(\mathcal{D}(M, X), \mathcal{S}(f, X)) \longrightarrow \pi_k \mathcal{O}(f, X)$ defined by $j_k[\omega] = [f \circ \omega]$ for a continuous map $\omega : (I^k, \partial I^k, 0) \rightarrow (\mathcal{D}(M), \mathcal{S}(f), \text{id}_M)$, and making commutative the following diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_k \mathcal{D}(M, X) & \xrightarrow{q} & \pi_k(\mathcal{D}(M, X), \mathcal{S}(f, X)) & \xrightarrow{\partial} & \pi_{k-1} \mathcal{S}(f, X) \longrightarrow \cdots \\ & & \searrow p & & \downarrow \cong j_k & \nearrow \partial \circ j_k^{-1} & \\ & & & & \pi_k \mathcal{O}(f, X), & & \end{array}$$

see for example [4, § 4.1, Theorem 4.1].

(3) Suppose either f has a critical point which is not a **nondegenerate local extremum** or M is a non-oriented surface. Then $\mathcal{S}_{\text{id}}(f)$ is contractible, $\pi_n \mathcal{O}(f) = \pi_n M$ for $n \geq 3$, $\pi_2 \mathcal{O}(f) = 0$, and for $\pi_1 \mathcal{O}(f)$ we have the following short exact sequence of fibration p :

$$1 \longrightarrow \pi_1 \mathcal{D}(M) \xrightarrow{p} \pi_1 \mathcal{O}(f) \xrightarrow{\partial \circ j_1^{-1}} \pi_0 \mathcal{S}'(f) \longrightarrow 1. \quad (1.4)$$

Moreover, $p(\pi_1 \mathcal{D}(M))$ is contained in the center of $\pi_1 \mathcal{O}(f)$.

(4) Suppose either $\chi(M) < 0$ or $X \neq \emptyset$. Then $\mathcal{D}_{\text{id}}(M, X)$ and $\mathcal{S}_{\text{id}}(f, X)$ are contractible, whence from the exact sequence of homotopy groups of the fibration (1.3) we get $\pi_k \mathcal{O}(f, X) = 0$ for $k \geq 2$, and that the boundary map

$$\partial \circ j_1^{-1} : \pi_1 \mathcal{O}(f, X) \longrightarrow \pi_0 \mathcal{S}'(f, X)$$

is an isomorphism.

Suppose M is orientable and differs from the sphere S^2 and the torus T^2 , and let $X = \partial M$. Then M and X satisfy assumptions of (4) of Theorem 1.3. Therefore from (1) of that theorem we get the following isomorphism

$$\pi_1 \mathcal{O}(f) \stackrel{(1)}{\cong} \pi_1 \mathcal{O}(f, \partial M) \stackrel{(4)}{\cong} \pi_0 \mathcal{S}'(f, \partial M).$$

A possible structure of $\pi_0 \mathcal{S}'(f, \partial M)$ for this case is completely described in [13].

However when M is a sphere or a torus the situation is more complicated, as $\pi_1 \mathcal{D}(M) \neq 0$ and from the short exact sequence (1.4) we get only that $\pi_1 \mathcal{O}(f)$ is an extension of $\pi_0 \mathcal{S}'(f)$ with $\pi_1 \mathcal{D}(M)$.

2. MAIN RESULT

Suppose $M = T^2$. Then it can easily be shown that for each $f \in \mathcal{F}(T^2)$ its KR-graph $\Gamma(f)$ is either a tree or has exactly one simple cycle. Moreover, $\pi_1 \mathcal{D}^{\text{id}} = \mathbb{Z}^2$, see [2, 3], and therefore the sequence (1.4) can be rewritten as follows:

$$1 \longrightarrow \mathbb{Z}^2 \xrightarrow{p} \pi_1 \mathcal{O}_f(f) \xrightarrow{\partial} \pi_0 \mathcal{S}'(f) \longrightarrow 1. \quad (2.1)$$

In [14] the authors studied the case when $\Gamma(f)$ is a tree and proved that under certain “triviality of $\mathcal{S}'(f)$ -action” assumptions on f the sequence (2.1) splits and we get an isomorphism $\pi_1 \mathcal{O}_f(f) \cong \pi_0 \mathcal{S}'(f) \times \mathbb{Z}^2$.

In the present paper we consider the situation when $\Gamma(f)$ has exactly one simple cycle Υ and under another “triviality of $\mathcal{S}'(f)$ -action” assumption describe the homotopy type of $\mathcal{O}_f(f)$ in terms of $\pi_0 \mathcal{S}'(f, C)$ for some regular component of some level-set of f , see Definition 2.3 and Theorem 2.6.

First we mention the following two simple lemmas which are left for the reader.

Lemma 2.1. *Let $f \in \mathcal{F}(T^2)$. Then the following conditions are equivalent:*

- (i) $\Gamma(f)$ is a tree;
- (ii) every point $z \in \Gamma(f)$ separates $\Gamma(f)$;
- (iii) every connected component of every level set of f separates T^2 .

Lemma 2.2. *Assume that $\Gamma(f)$ has exactly one simple cycle Υ . Let also $z \in \Gamma(f)$ be any point belonging to some open edge of $\Gamma(f)$ and C be the corresponding regular component of certain level set $f^{-1}(c)$ of f . Then the following conditions are equivalent:*

- (a) $z \in \Upsilon$;
- (b) z does not separate $\Gamma(f)$;
- (c) C does not separate T^2 .

Thus if $\Gamma(f)$ is not a tree, then there exists a connected component C of some level set of f that does not separate T^2 , and this curve corresponds to some point z on an open edge of cycle Υ .

For simplicity we will fix once and for all such $f \in \mathcal{F}(T^2)$ and C , and use the following notation:

$$\begin{aligned} \mathcal{D}^{\text{id}} &:= \mathcal{D}_{\text{id}}(T^2), & \mathcal{O} &:= \mathcal{O}_f(f), & \mathcal{S} &:= \mathcal{S}'(f), & \mathcal{S}^{\text{id}} &:= \mathcal{S}_{\text{id}}(T^2), \\ \mathcal{D}_C^{\text{id}} &:= \mathcal{D}_{\text{id}}(T^2, C), & \mathcal{O}_C &:= \mathcal{O}_f(f, C), & \mathcal{S}_C &:= \mathcal{S}'(f, C), & \mathcal{S}_C^{\text{id}} &:= \mathcal{S}_{\text{id}}(f, C) \end{aligned}$$

Let also $h \in \mathcal{S}$, so $f \circ h = f$ and h is isotopic to id_{T^2} . Then $h(f^{-1}(c)) = f^{-1}(c)$, and therefore h interchanges connected components of $f^{-1}(c)$. In particular, $h(C)$ is also a connected component of $f^{-1}(c)$. However, in general, $h(C)$ does not coincide with C , see Figure 2.1.

Definition 2.3. *We will say that \mathcal{S} trivially acts on C if $h(C) = C$ for all $h \in \mathcal{S}$.*

Remark 2.4. Emphasize that the above definition only require that $h(C) = C$ for all diffeomorphisms h that preserve f and are isotopic to id_{T^2} . We do not put any assumptions on diffeomorphisms that are not isotopic to id_{T^2} .

Remark 2.5. It is easy to see that if C_1 is another non-separating regular component of some level-set $f^{-1}(c_1)$, then \mathcal{S} trivially acts on C if and only if \mathcal{S} trivially acts on C_1 .

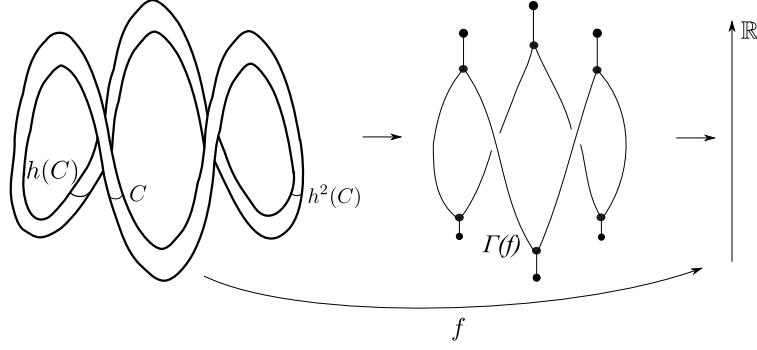


FIGURE 2.1.

Theorem 2.6. *Let $f \in \mathcal{F}(T^2)$ be such that $\Gamma(f)$ has exactly one cycle, and C be a regular connected component of certain level set $f^{-1}(c)$ of f which does not separate T^2 . Suppose \mathcal{S} trivially acts on C . Then there is a homotopy equivalence*

$$\mathcal{O} \simeq \mathcal{O}_C \times S^1.$$

In particular, we have the following isomorphisms:

$$\pi_1 \mathcal{O} \cong \pi_1 \mathcal{O}_C \times \mathbb{Z} \xrightarrow{j \times \text{id}_{\mathbb{Z}}} \pi_0 \mathcal{S}_C \times \mathbb{Z}.$$

The proof of this theorem will be given in § 4.

3. PRELIMINARIES

3.1. Algebraic lemma.

Lemma 3.2. *Let L, M, Q, S be four groups. Suppose there exists a short exact sequence*

$$1 \rightarrow L \times M \xrightarrow{q} T \xrightarrow{\partial} S \rightarrow 1 \quad (3.1)$$

and a homomorphism $\varphi : T \rightarrow L \times 1$ such that

- $\varphi \circ q : L \times 1 \rightarrow L \times 1$ is the identity map and
- $q(1 \times M) \subset \ker(\varphi)$.

Then we have the following exact sequence:

$$1 \rightarrow 1 \times M \xrightarrow{q} \ker(\varphi) \xrightarrow{\partial} S \rightarrow 1.$$

Proof. It suffices to prove that

- 1) $\partial(\ker \varphi) = S$,
- 2) $\varphi(1 \times M) = \ker \varphi \cap \ker \partial$.

1) Let $s \in S$. We have to find $b \in \ker \varphi$ such that $\partial(b) = s$. Since $\partial(T) = S$, there exists $t \in T$ such that $\partial(t) = s$. Put $\hat{t} = q(\varphi(t))$ and $b = t\hat{t}^{-1}$. Then

$$\varphi(\hat{t}) = \varphi \circ q \circ \varphi(t) = \varphi(t), \quad \partial(\hat{t}) = \partial \circ q \circ \varphi(t) = 1.$$

Hence $b = t\hat{t}^{-1} \in \ker \varphi$ and

$$\partial(b) = \partial(t) \partial(\hat{t})^{-1} = \partial(t) = s.$$

2) Let $a \in \ker \varphi \cap \ker \partial$. We should find $m \in M$ such that $q(1, m) = a$.

As $a \in \ker \partial = q(L \times M)$, so there exist $(l, m) \in L \times M$ such that $q(l, m) = a$. But $q(1, m) \in q(1 \times M) \subset \ker \varphi$, whence

$$(1, 1) = \varphi(a) = \varphi(q(l, m)) = \varphi(q(l, 1)) \cdot \varphi(q(1, m)) = \varphi(q(l, 1)) = (l, 1).$$

Hence $l = 1$, and so $a = q(1, m) \in q(1 \times M)$. \square

3.3. Isotopies of T^2 fixed on a curve. We will need the following general lemma claiming that if a diffeomorphism h of T^2 is fixed on a non-separating simple closed curve C and is isotopic to id_{T^2} , then an isotopy between h and id_{T^2} can be made fixed on C .

Lemma 3.4. *Let $C \subset T^2$ be a not null-homotopic smooth simple closed curve. Then*

$$\mathcal{D}_{\text{id}}(T^2, C) = \mathcal{D}_{\text{id}}(T^2) \cap \mathcal{D}(T^2, C). \quad (3.2)$$

Proof. The inclusion $\mathcal{D}_{\text{id}}(T^2, C) \subset \mathcal{D}_{\text{id}}(T^2) \cap \mathcal{D}(T^2, C)$ is evident. Therefore we have to establish the inverse one.

Let $h \in \mathcal{D}_{\text{id}}(T^2) \cap \mathcal{D}(T^2, C)$, so h is fixed on C and is isotopic to id_{T^2} . We have to prove that $h \in \mathcal{D}_{\text{id}}(T^2, C)$, i.e. it is isotopic to id_{T^2} via an isotopy fixed on C .

Let $C_1 \subset T^2$ be a simple closed curve isotopic to C and disjoint from C , and $\tau : T^2 \rightarrow T^2$ be a Dehn twist along C_1 fixed on C . Cut the torus T^2 along C and denote the resulting cylinder by Q .

Notice that the restrictions $h|_Q, \tau|_Q : Q \rightarrow Q$ are fixed on ∂Q . It is well-known that the isotopy class $\tau|_Q$ generates the group $\pi_0 \mathcal{D}(Q, \partial Q) \cong \mathbb{Z}$. Hence there exists $n \in \mathbb{Z}$ such that $h|_Q$ is isotopic to $\tau^n|_Q$ relatively to ∂Q . This isotopy induces an isotopy between h and τ^n fixed on C .

By assumption h is isotopic to id_{T^2} , while τ^n is isotopic to id_{T^2} only for $n = 0$. Hence h is isotopic to $\tau^0 = \text{id}_{T^2}$ via an isotopy fixed of C . \square

3.5. Smooth shifts along trajectories of a flow. Let $\mathbf{F} : M \times \mathbb{R} \rightarrow M$ be a smooth flow on a manifold M . Then for every smooth function $\alpha : M \rightarrow \mathbb{R}$ one can define the following map $\mathbf{F}_\alpha : T^2 \rightarrow \mathbb{R}$ by the formula:

$$\mathbf{F}_\alpha(z) = \mathbf{F}(z, \alpha(z)), \quad z \in M. \quad (3.3)$$

Lemma 3.6. *If \mathbf{F}_α is a diffeomorphism then for each $t \in [0, 1]$ the map*

$$\mathbf{F}_{t\alpha} : M \rightarrow M, \quad \mathbf{F}_{t\alpha}(z) = \mathbf{F}(z, t\alpha(z))$$

is a diffeomorphism as well. In particular, $\{\mathbf{F}_{t\alpha}\}_{t \in I}$ is an isotopy between $\text{id}_M = \mathbf{F}_0$ and \mathbf{F}_α .

3.7. Some constructions associated with f . In the sequel we will regard the circle S^1 and the torus T^2 as the corresponding factor-groups \mathbb{R}/\mathbb{Z} and $\mathbb{R}^2/\mathbb{Z}^2$. Let $e = (0, 0) \in T^2$ be the unit of T^2 . We will always assume that e is a base point for all homotopy groups related with T^2 and its subsets. For $\varepsilon \in (0, 0.5)$ let

$$J_\varepsilon = (-\varepsilon, \varepsilon) \subset S^1$$

be an open ε -neighbourhood of $0 \in S^1$.

Let $f \in \mathcal{F}(T^2)$ be a function such that its KR-graph $\Gamma(f)$ has only one cycle, and let C be a regular connected component of certain level set of f not separating T^2 . For this situation we will now define several constructions “adopted” with f .

Special coordinates. Since C is non-separating and if a regular component of $f^{-1}(c)$, one can assume (by a proper choice of coordinates on T^2) that the following two conditions hold:

- (a) $C = 0 \times S^1 \subset \mathbb{R}^2/\mathbb{Z}^2 \equiv T^2$;
- (b) there exists $\varepsilon > 0$ such that for all $t \in J_\varepsilon = (-\varepsilon, \varepsilon)$ the curve $t \times S^1$ is a regular connected component of some level set of f .

It is convenient to regard C as a *meridian* of T^2 . Let $C' = S^1 \times 0$ be the corresponding *parallel*. Then $C' \cap C = e$, see Figure 3.1. Consider also the following loops $\lambda, \mu : I \rightarrow T^2$ defined by

$$\lambda(t) = (t, 0), \quad \mu(t) = (0, t). \quad (3.4)$$

They represent the homotopy classes of C' and C in $\pi_1 T^2$ respectively.

Let us also mention that C is a *subgroup* of the group T^2 . Therefore $\pi_1(T^2, C)$ has a natural groups structure.

Let $k : C \hookrightarrow T^2$ be the inclusion map. Then the corresponding homomorphism $k : \pi_1 C \rightarrow \pi_1 T^2$ is injective. Since C is also connected, i.e. $\pi_0 C = \{1\}$, we get the following short exact sequence of homotopy groups of the pair (T^2, C) :

$$1 \longrightarrow \pi_1 C \xrightarrow{k} \pi_1 T^2 \xrightarrow{r} \pi_1(T^2, C) \longrightarrow 1. \quad (3.5)$$

As $\pi_1 C \cong \mathbb{Z}$ and $\pi_1 T^2 \cong \mathbb{Z}^2$, it follows that $\pi_1(T^2, C) \cong \mathbb{Z}$ and this group is generated by the image $r[\lambda]$ of the homotopy class of the parallel λ . In particular, there exists a section

$$s : \pi_1(T^2, C) \longrightarrow \pi_1 T^2 \quad (3.6)$$

such that $r \circ s[\lambda] = [\lambda]$, so $r \circ s$ is the identity map of $\pi_1(T^2, C)$.

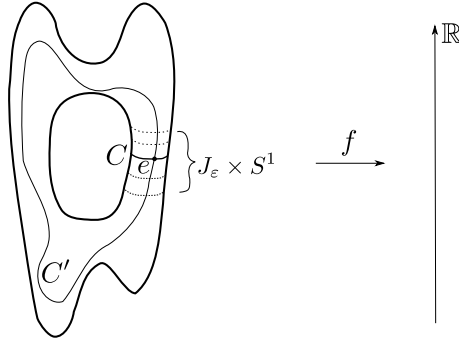


FIGURE 3.1.

An inclusion $\xi : T^2 \subset \mathcal{D}^{\text{id}}$. Notice that T^2 is a connected Lie group. Therefore it acts on itself by smooth left translations. This yields the following embedding $\xi : T^2 \hookrightarrow \mathcal{D}^{\text{id}}$: if $(a, b) \in T^2$, then $\xi(a, b) : T^2 \rightarrow T^2$ is a diffeomorphism given by the formula:

$$\xi(a, b)(x, y) = (x + a \bmod 1, y + b \bmod 1). \quad (3.7)$$

It is well known that ξ is a homotopy equivalence, see e.g. [3].

Notice also that ξ yields the following map

$$\xi : C((I, \partial I), (T^2, e)) \longrightarrow C((I, \partial I), (\mathcal{D}^{\text{id}}, \text{id}_{T^2})) \quad (3.8)$$

between the *spaces of loops* defined as follows: if $\omega : (I, \partial I) \longrightarrow (\mathcal{D}^{\text{id}}, \text{id}_{T^2})$ is a continuous map, then

$$\xi(\omega) = \xi \circ \omega : I \longrightarrow \mathcal{D}^{\text{id}}.$$

It is well known that this map is continuous with respect to compact open topologies. Moreover the corresponding map between the path components is just the homomorphism of fundamental groups:

$$\xi : \pi_1 T^2 \longrightarrow \pi_1 \mathcal{D}^{\text{id}}. \quad (3.9)$$

Since ξ is a homotopy equivalence, the homomorphism (3.9) is in fact an isomorphism.

To simplify notation we denoted all these maps with the same letter ξ . However this will never lead to confusion.

Isotopies \mathbf{L} and \mathbf{M} . Let

$$\mathbf{L} = \xi(\lambda), \quad \mathbf{M} = \xi(\mu) \quad (3.10)$$

be the images of the loops λ and μ in \mathcal{D}^{id} under the map Eq. (3.8). Evidently, they can be regarded as isotopies $\mathbf{L}, \mathbf{M} : T^2 \times [0, 1] \rightarrow T^2$ defined by

$$\mathbf{L}(x, y, t) = (x + t \bmod 1, y), \quad \mathbf{M}(x, y, t) = (x, y + t \bmod 1), \quad (3.11)$$

for $x \in C'$, $y \in C$, and $t \in [0, 1]$. Geometrically, \mathbf{L} is a “rotation” of the torus along its parallels and \mathbf{M} is a rotation along its meridians.

Denote by \mathcal{L} and \mathcal{M} the subgroups of $\pi_1 \mathcal{D}^{\text{id}}$ generated by loops \mathbf{L} and \mathbf{M} respectively. Since $\xi : T^2 \rightarrow \mathcal{D}^{\text{id}}$ is a homotopy equivalence, and the loops λ and μ freely generate $\pi_1 T^2$, it follows that \mathcal{L} and \mathcal{M} are commuting free cyclic groups, and so we get an isomorphism:

$$\pi_1 \mathcal{D}^{\text{id}} \cong \mathcal{L} \times \mathcal{M}.$$

Also notice that \mathbf{L} and \mathbf{M} can be also regarded as *flows* $\mathbf{L}, \mathbf{M} : T^2 \times \mathbb{R} \rightarrow T^2$ defined by the same formulas Eq. (3.11) for $(x, y, t) \in T^2 \times \mathbb{R}$. All orbits of the *flows* \mathbf{L} and \mathbf{M} are periodic of period 1. We will denoted these flows by the same letters as the corresponding *loops* (3.10), however this will never lead to confusion.

A flow \mathbf{F} . As T^2 is an orientable surface, there exists a flow $\mathbf{F} : T^2 \times \mathbb{R} \rightarrow T^2$ having the following properties, see e.g. [9, Lemma 5.1]:

- 1) a point $z \in T^2$ is fixed for \mathbf{F} if and only if z is a critical point of f ;
- 2) f is constant along orbits of \mathbf{F} , that is $f(z) = f(\mathbf{F}(z, t))$ for all $z \in T^2$ and $t \in \mathbb{R}$.

It follows that every critical point of f and every regular components of every level set of f is an orbit of \mathbf{F} .

In particular, each curve $t \times S^1$ for $t \in J_\varepsilon$ is an orbit of \mathbf{F} . On the other hand, this curve is also an orbit of the flow \mathbf{M} . Therefore, we can always choose \mathbf{F} so that

$$\mathbf{M}(x, y, t) = \mathbf{F}(x, y, t), \quad (x, y, t) \in J_\varepsilon \times S^1 \times \mathbb{R}. \quad (3.12)$$

Lemma 3.8. [9, Lemma 5.1]. *Suppose a flow $\mathbf{F} : T^2 \times \mathbb{R} \rightarrow T^2$ satisfies the above conditions 1) and 2) and let $h \in \mathcal{S}(f)$. Then $h \in \mathcal{S}_{\text{id}}(f)$ if and only if there exists a C^∞ function $\alpha : T^2 \rightarrow \mathbb{R}$ such that $h = \mathbf{F}_\alpha$, see Eq. (3.3). Such a function is unique and the family of maps $\{\mathbf{F}_{t\alpha}\}_{t \in I}$ constitute an isotopy between id_M and h .* \square

4. PROOF OF THEOREM 2.6

Let $f \in \mathcal{F}(T^2)$ be such that its KR-graph $\Gamma(f)$ has only one cycle, and let C be a non-separating regular connected component of certain level set of f . Assume also that \mathcal{S} trivially acts of C . We have to prove that there exists a homotopy equivalence $\mathcal{O}_C \times S^1 \simeq \mathcal{O}$.

By (3) and (4) of Theorem 1.3 the orbits \mathcal{O} and \mathcal{O}_C are aspherical, as well as S^1 , i.e. their homotopy groups π_k vanish for $k \geq 2$. Therefore, by Whitehead Theorem [4, § 4.1, Theorem 4.5], it suffices to show that there exists an isomorphism $\pi_1 \mathcal{O}_C \times \pi_1 S^1 \cong \pi_1 \mathcal{O}$. Such an isomorphism will induce a required homotopy equivalence.

Moreover, due to (2) of Theorem 1.3 we have isomorphisms:

$$\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \cong \pi_1 \mathcal{O}_C, \quad \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}) \cong \pi_1 \mathcal{O}.$$

Therefore it remains to find the following isomorphism:

$$\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \times \pi_1 S^1 \cong \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}). \quad (4.1)$$

Notice that every smooth function $f : T^2 \rightarrow \mathbb{R}$ always have critical points being not local extremes, since otherwise T^2 would be diffeomorphic with a 2-sphere S^2 . Therefore by (3) and (4) of Theorem 2.6 the spaces \mathcal{S} , \mathcal{S}_C , and $\mathcal{D}_C^{\text{id}}$ are contractible. Moreover, as noted above, \mathcal{D}^{id} is homotopy equivalent to T^2 .

Let $i : (\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \subset (\mathcal{D}^{\text{id}}, \mathcal{S})$ be the inclusion map. It yields a morphism between the exact sequences of homotopy groups of these pairs. The non-trivial part of this morphism is contained in the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 & \longrightarrow & \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) & \xrightarrow{\partial_C} & \pi_0 \mathcal{S}_C & \longrightarrow & 1 \\ & & \downarrow & & i_1 \downarrow & & i_0 \downarrow & & \\ 1 & \longrightarrow & \pi_1 \mathcal{D}^{\text{id}} & \xrightarrow{q} & \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}) & \xrightarrow{\partial} & \pi_0 \mathcal{S} & \longrightarrow & 1 \end{array} \quad (4.2)$$

The proof of Theorem 2.6 is based on the following two Propositions 4.1 and 4.3 below.

Proposition 4.1. *Under assumptions of Theorem 2.6 there exists an epimorphism*

$$\varphi : \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}) \longrightarrow \mathcal{L}$$

such that

- 1) φ is a left inverse for q , that is $\varphi \circ q = \text{id}_{\mathcal{L}}$
- 2) $q(\mathcal{M}) \subset \ker \varphi$
- 3) $i_1(\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)) \subset \ker \varphi$.

Corollary 4.2. a) *The map $\theta : \ker \varphi \times \mathcal{L} \longrightarrow \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S})$ defined by $\theta(\omega, l) = \omega \cdot q(l)$ for $(\omega, l) \in \ker \varphi \times \mathcal{L}$, is a groups isomorphism.*

b) *The following sequence is exact: $1 \longrightarrow \mathcal{M} \xrightarrow{q} \ker \varphi \xrightarrow{\partial} \pi_0 \mathcal{S} \longrightarrow 1$.*

Proof. Statement a) follows from 1) of Proposition 4.1 and the fact that $q(\mathcal{L})$ is contained in the center of $\pi_1(\mathcal{D}^{\text{id}}, \mathcal{S})$, see (3) of Theorem 2.6. Statement b) is a direct consequence of statements 1) and 2) of Proposition 4.1 and Lemma 3.2 applied to the lower exact sequence of Eq. (4.2). We leave the details for the reader. \square

Due to 2) and 3) of Proposition 4.1 and 3) of Corollary 4.2 we see that diagram Eq. (4.2) reduces to the following one:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & 1 & \longrightarrow & \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) & \xrightarrow[\cong]{\partial_C} & \pi_0 \mathcal{S}_C \longrightarrow 1 \\
 & & \downarrow & & i_1 \downarrow & & i_0 \downarrow \\
 1 & \longrightarrow & \mathcal{M} & \xrightarrow{q} & \ker \varphi & \xrightarrow{\partial} & \pi_0 \mathcal{S} \longrightarrow 1
 \end{array} \tag{4.3}$$

Thus to complete Theorem 2.6 it suffices to prove that the middle vertical arrow, i_1 , in Eq. (4.3) is an isomorphism between $\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)$ and $\ker \varphi$. As $\mathcal{L} \cong \mathbb{Z} \cong \pi_1 S^1$ we will then get the required isomorphism

$$\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \times \mathbb{Z} \cong \ker \varphi \times \mathcal{L} \xrightarrow{\theta} \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}).$$

To show that $i_1 : \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \rightarrow \ker \varphi$ is an isomorphism notice that ∂_C is also an isomorphism by (4) of Theorem 1.3. Therefore from the latter diagram Eq. (4.3) we get the following one:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \ker i_0 & \longrightarrow & \pi_0 \mathcal{S}_C & \xrightarrow{i_0} & \pi_0 \mathcal{S} \\
 & & \downarrow & & i_1 \circ \partial_C^{-1} \downarrow & & \parallel \\
 1 & \longrightarrow & q(\mathcal{M}) & \longrightarrow & \ker \varphi & \xrightarrow{\partial} & \pi_0 \mathcal{S} \longrightarrow 1
 \end{array}$$

Proposition 4.3. *Homomorphism $i_0 : \pi_0 \mathcal{S}_C \rightarrow \pi_0 \mathcal{S}$ is surjective, and the induced map*

$$i_1 \circ \partial_C^{-1} : \ker i_0 \longrightarrow q(\mathcal{M})$$

is an isomorphism.

In other words, Proposition 4.3 claims that we have the following morphism between short exact sequences:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \ker i_0 & \longrightarrow & \pi_0 \mathcal{S}_C & \xrightarrow{i_0} & \pi_0 \mathcal{S} \longrightarrow 1 \\
 & & \cong \downarrow & & i_1 \circ \partial_C^{-1} \downarrow & & \parallel \\
 1 & \longrightarrow & q(\mathcal{M}) & \longrightarrow & \ker \varphi & \xrightarrow{\partial} & \pi_0 \mathcal{S} \longrightarrow 1
 \end{array}$$

Since left and right vertical arrows are isomorphisms, it will follow from five lemma, [4, § 2.1], that $i_1 \circ \partial_C^{-1}$ is an isomorphism as well. This completes Theorem 2.6 modulo Propositions 4.1 and 4.3. The next two sections are devoted to the proof of those propositions.

Remark 4.4. It easily follows from statement a) of Corollary 4.2 that the map $\kappa : \mathcal{O}_C \times S^1 \rightarrow \mathcal{O}$ defined by:

$$\kappa(g, t) = g \circ \mathbf{L}_t \tag{4.4}$$

is a homotopy equivalence. We leave the details for the reader.

5. PROOF OF PROPOSITION 4.1

Existence of φ is guaranteed by statement (e) of the following lemma.

Lemma 5.1. *Suppose \mathcal{S} trivially acts of C . Then there is a commutative diagram:*

$$\begin{array}{ccccc}
 \pi_1 \mathcal{D}^{\text{id}} & \xleftarrow[\cong]{\xi} & \pi_1 T^2 & & \\
 q \downarrow & & \uparrow s \downarrow r & & \\
 \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) & \xrightarrow{i_1} & \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}) & \xrightarrow{\zeta} & \pi_1(T^2, C) \cong \mathbb{Z}
 \end{array} \tag{5.1}$$

in which

- (a) $r \circ s$ is the identity isomorphism of $\pi_1(T^2, C)$;
- (b) $\xi \circ s$ is an isomorphism of $\pi_1(T^2, C)$ onto \mathcal{L} ;
- (c) ζ is surjective;
- (d) $q(\mathcal{M}) \subset \ker \zeta$;
- (e) $i_1(\pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)) \subset \ker \zeta$.
- (f) The following composition

$$\varphi = \xi \circ s \circ \zeta : \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \longrightarrow \mathcal{L}$$

satisfies the statement of Proposition 4.1.

Proof. We need only to define the map ζ , since q appears in Eq. (1.2), and r , s , and ξ are described in §3.7.

Let $\omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2})$ be a continuous map. In particular, $\omega(1)$ belongs to \mathcal{S} . By assumption \mathcal{S} trivially acts on C , whence $\omega(1)(C) = C$. Consider the following path

$$\omega_e : I \rightarrow T^2, \quad \omega_e(t) = \omega(t)(e).$$

Then $\omega_e(0) = e$ and $\omega_e(1) \in C$. Therefore $\omega_e \in C((I, \partial I, 0), (T^2, C, e))$ and we put by definition

$$\zeta(\omega) = \omega_e.$$

The map $\zeta : \pi_1(\mathcal{D}^{\text{id}}, \mathcal{S}) \rightarrow \pi_1(T^2, C)$ in Eq. (5.1) is the induced mapping of the corresponding sets of homotopy classes. It is easy to verify that ζ is in fact a group homomorphism.

Commutativity of diagram Eq. (5.1). Notice that the groups in right rectangle of Eq. (5.1) are just the sets of path components of the corresponding spaces from the following diagram:

$$\begin{array}{ccc}
 C((I, \partial I), (\mathcal{D}^{\text{id}}, \text{id}_{T^2})) & \xleftarrow{\xi} & C((I, \partial I), (T^2, e)) \\
 q \downarrow & & \downarrow r \\
 C((I, \partial I, 0), (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2})) & \xrightarrow{\zeta} & C((I, \partial I, 0), (T^2, C, e))
 \end{array} \tag{5.2}$$

Notice that the maps q and r here are just natural inclusions. It suffices to prove commutativity of diagram (5.2).

Let $\omega = (\alpha, \beta) : (I, \partial I) \longrightarrow (T^2, e)$ be a representative of some loop in $\pi_1 T^2$, where α and β are coordinate functions of ω . Then

$$q \circ \xi(\omega)(t)(x, y) = (x + \alpha(t) \bmod 1, y + \beta(t) \bmod 1).$$

whence

$$\zeta \circ q \circ \xi(\omega)(t) = q \circ \xi(\omega)(t)(0, 0) = (\alpha(t), \beta(t)) = \omega(t).$$

But $r(\omega)(t) = \omega(t)$ as well, whence $\zeta \circ q \circ \xi(\omega) = r(\omega)$. Thus diagram Eq. (5.1) is commutative.

Property (a) is already established, see remark just after Eq. (3.6).

Property (b). Since $\xi \circ s(r[\lambda]) = \xi[\lambda] = \mathbf{L}$ and \mathcal{L} is freely generated by \mathbf{L} , it follows that $\xi \circ s$ isomorphically maps $\pi_1(T^2, C)$ onto \mathcal{L} .

Property (c). By commutativity of Eq. (5.1) we have that $r[\lambda] = \zeta \circ q \circ \xi[\lambda]$. But $r[\lambda]$ generates $\pi_1(T^2, \lambda)$, whence ζ is surjective.

Property (d). As \mathbf{M} generates \mathcal{M} , it suffices to show that $\zeta \circ q(\mathbf{M}) = 0$. We have that $[\mu] \in k(\pi_1 C)$. Therefore it follows from exactness of sequence (3.5) that $r[\mu] = 0 \in \pi_1(T^2, C)$. Hence

$$\zeta \circ q[\mathbf{M}] = \zeta \circ q \circ \xi[\mu] = r[\mu] = 0.$$

Thus $q(\mathcal{M}) \subset \ker \varphi$;

Property (e). Let $\alpha \in \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)$ and $\omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}_C^{\text{id}}, \mathcal{S}_C, \text{id}_{T^2})$ a path representing α . Then $i_1(\alpha)$ is represented by the homotopy class of the map

$$i \circ \omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2}).$$

By assumption $\omega(t) \in \mathcal{D}_C^{\text{id}}$, i.e. it is fixed on C for all $t \in I$. In particular, since $e \in C$, we get that

$$\zeta(\omega)(t) = \omega_e(t) = \omega(t)(e) = e.$$

Thus $\zeta(\omega) : I \rightarrow T^2$ is a constant map, and so it represents a unit element of $\pi_1(T^2, C')$. Therefore $i_1(\alpha) \in \ker \zeta$.

Property (f). By (b) and (c) $\xi \circ s$ is an isomorphism, and ζ is surjective. Hence $\varphi = \xi \circ s \circ \zeta$ is surjective as well, and $\ker \varphi = \ker \zeta$. Therefore statements 2) and 3) of Proposition 4.1 follow from (d) and (e) respectively.

To prove 1) notice that

$$\varphi \circ q(\mathbf{L}) = (\xi \circ s \circ \zeta) \circ q(\xi[\lambda]) = \xi \circ s \circ (\zeta \circ q \circ \xi)[\lambda] = \xi \circ s \circ r[\lambda] = \xi[\lambda] = \mathbf{L}.$$

Hence $\varphi \circ q = \text{id}_{\mathcal{L}}$. Lemma 5.1 and Proposition 4.1 are completed. \square

6. PROOF OF PROPOSITION 4.3

6.1. **Image of i_0 .** Surjectivity of i_0 is guaranteed by the following lemma.

Lemma 6.2. *Suppose \mathcal{S} trivially acts on C , i.e. $h(C) = C$ for all $h \in \mathcal{S}$. Then the induced homomorphism $i_0 : \pi_0 \mathcal{S}_C \rightarrow \pi_0 \mathcal{S}$ is surjective.*

Proof. We should prove that each $h \in \mathcal{S} \equiv \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(T^2)$ is isotopic in \mathcal{S} to a diffeomorphism g belonging to $\mathcal{S}_C \equiv \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(T^2, C)$.

Indeed, by assumption $h(C) = C$. Hence h is isotopic in \mathcal{S} to a diffeomorphism g fixed on C , i.e. $g \in \mathcal{S} \cap \mathcal{D}(T^2, C)$. But due to Lemma 3.4

$$\mathcal{S} \cap \mathcal{D}(T^2, C) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(T^2) \cap \mathcal{D}(T^2, C) \stackrel{(3.2)}{=} \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(T^2, C) = \mathcal{S}_C,$$

whence $g \in \mathcal{S}_C$. Thus i_0 is epimorphism. \square

6.3. Kernel of i_0 . Notice that the kernel of $i_0 : \pi_0 \mathcal{S}_C \rightarrow \pi_0 \mathcal{S}$ consists of isotopy classes of diffeomorphisms in \mathcal{S}_C isotopic to id_{T^2} by f -preserving isotopy, however such an isotopy should not necessarily be fixed on C . In other words, if we denote

$$\mathcal{K} := \mathcal{S}^{\text{id}} \cap \mathcal{D}_C^{\text{id}} = \mathcal{S}_{\text{id}}(f) \cap \mathcal{D}(T^2, C),$$

then

$$\ker i_0 = \pi_0 \mathcal{K}. \quad (6.1)$$

Also notice that $\mathcal{S}_C^{\text{id}}$ is the identity path component of \mathcal{K} , whence

$$\ker i_0 = \pi_0 \mathcal{K} = \mathcal{K} / \mathcal{S}_C^{\text{id}}.$$

For the proof of Proposition 4.3 we will first establish in Lemma 6.4 that $\ker i_0 \cong \mathbb{Z}$ and then show in Lemma 6.6 that $i_1 \circ \partial_C^{-1}$ yields an isomorphism of $\ker i_0$ onto \mathcal{M} .

Since $\mathcal{K} := \mathcal{S}^{\text{id}} \cap \mathcal{D}_C^{\text{id}} \subset \mathcal{S}^{\text{id}}$, it follows from Lemma 3.8 that for every $h \in \mathcal{K}$ there exists a unique smooth function $\delta \in C^\infty(T^2)$ such that $h = \mathbf{F}_\delta$.

Lemma 6.4. *Suppose the flow \mathbf{F} satisfies the relation Eq. (3.12). Then for each $h = \mathbf{F}_\delta \in \mathcal{K}$ the function δ is constant on C and its value on C is integer. Define a map $\eta : \mathcal{K} \rightarrow \mathbb{Z}$ by*

$$\eta(h) = \delta|_C,$$

for $h = \mathbf{F}_\delta \in \mathcal{K}$. Then η is a surjective homomorphism with $\ker \eta = \mathcal{S}_C^{\text{id}}$. In particular η yields an isomorphism $\ker i_0 = \pi_0 \mathcal{K} = \mathcal{K} / \mathcal{S}_C^{\text{id}} \cong \mathbb{Z}$.

Proof. We will regard \mathbf{M} as a flow on T^2 . Notice that C is a closed trajectory of both flows \mathbf{F} and \mathbf{M} and its period with respect to \mathbf{M} is equal to 1. Therefore the relation Eq. (3.12) implies that the period of C with respect to \mathbf{F} also equals 1.

Since $h \in \mathcal{K} = \mathcal{S}^{\text{id}} \cap \mathcal{D}_C^{\text{id}}$ is fixed on C , that is $y = h(y) = \mathbf{F}(y, \delta(y))$ for all $y \in C$, it follows that the value of $\delta(y)$ a multiple of the period $\theta = 1$. Thus for every $y \in C$ there exists $n_y \in \mathbb{Z}$ such that $\delta(y) = n_y$. But δ is continuous, whence the mapping $y \mapsto n_y = \delta(y)$ is a continuous function $C \rightarrow \mathbb{Z}$. Therefore this function is constant, i.e. $\delta|_C = n$ for some $n \in \mathbb{Z}$.

We should prove that η has the desired properties.

Step 1. *η is a homomorphism.*

Let $h_i = \mathbf{F}_{\delta_i} \in \mathcal{K}$ for $i = 0, 1$ such that $\delta_i|_C = n_i$ for certain $n_i \in \mathbb{Z}$. Define the function $\delta = \delta_1 \circ h_0 + \delta_0$. Since h_0 is fixed on C , we have that for each $z \in C$

$$\delta(z) = \delta_1 \circ h_0(z) + \delta_0(z) = \delta_1(z) + \delta_0(z) = n_1 + n_0.$$

Moreover, by [8, Eq. (8)],

$$h_1 \circ h_0 = \mathbf{F}_\delta,$$

whence

$$\eta(h_1 \circ h_0) = \delta|_C = n_1 + n_0 = \delta_1|_C + \delta_0|_C = \eta(h_1) + \eta(h_0),$$

and so η is a homomorphism.

Step 2. *η is surjective.*

It suffices to construct $g \in \mathcal{K}$ with $\eta(g) = -1$. The choice of -1 will simplify further exposition, however we could also construct g satisfying $\eta(g) = +1$.

Let J_ε be the same as in Eq. (3.12), and $\beta : J_\varepsilon \rightarrow [0, 1]$ be a C^∞ -function such that

$$\beta(z) = \begin{cases} -1, & |z| < \varepsilon/3, \\ 0 & |z| > 2\varepsilon/3. \end{cases}$$

Define another function $\sigma : T^2 \rightarrow \mathbb{R}$ by

$$\sigma(x, y) = \begin{cases} \beta(x), & x \in J_\varepsilon, \\ 0, & x \in S^1 \setminus J_\varepsilon, \end{cases}$$

and let $g = \mathbf{F}_\sigma$, so g is a map $T^2 \rightarrow T^2$ defined by

$$g(x, y) = \mathbf{F}(x, y, \sigma(x, y)).$$

Since $\sigma = 0$ outside $J_\varepsilon \times S^1$, it follows from Eq. (3.12) that $g = \mathbf{M}_\sigma$, i.e.

$$g(x, y) = \mathbf{M}(x, y, \sigma(x, y)). \quad (6.2)$$

Moreover, as periods of all points with respect to \mathbf{M} are equal to 1, we can also write $g = \mathbf{M}_{\sigma+n}$ for any $n \in \mathbb{Z}$, i.e.

$$g(x, y) = \mathbf{M}(x, y, \sigma(x, y) + n).$$

Thus

$$g = \mathbf{F}_\sigma = \mathbf{M}_\sigma = \mathbf{M}_{\sigma+n}$$

for all $n \in \mathbb{Z}$. The following properties of g can easily be verified:

- (i) g is fixed on $J_{\varepsilon/3} \times S^1$ and outside $J_\varepsilon \times S^1$;
- (ii) $g \in \mathcal{S}_{\text{id}}(f)$ by Lemma 3.8, since $g = \mathbf{F}_\sigma$;
- (iii) the following family of maps $\mathbf{G}_t = \mathbf{M}_{t(\sigma+1)}$, $t \in I$, is an isotopy between

$$\mathbf{G}_0 = \mathbf{M}_0 = \text{id}_{T^2}, \quad \mathbf{G}_1 = \mathbf{M}_{\sigma+1} = g, \quad (6.3)$$

see Figure 6.1. This isotopy is fixed on C , as $t(\sigma+1) = 0$ on C . Hence $g \in \mathcal{D}_C^{\text{id}}$ as well.

Thus by (ii) and (iii) $g \in \mathcal{S}(f) \cap \mathcal{D}_C^{\text{id}} = \mathcal{K}$. It remains to note that $\eta(g) = \sigma|_C = -1$, and so η is surjective.

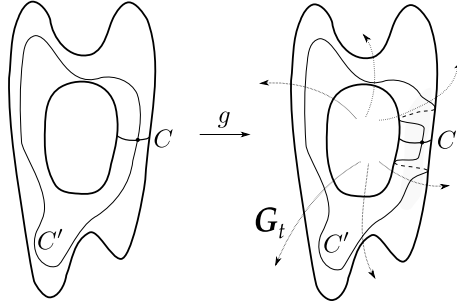


FIGURE 6.1. Diffeomorphism g and the isotopy \mathbf{G}_t

Step 3. $\ker \eta = \mathcal{S}_C^{\text{id}}$.

Suppose $h = \mathbf{F}_\delta \in \ker \eta$, so $\eta(h) = \delta|_C = 0$. Then by Lemma 3.8 an isotopy between h and id_{T^2} can be given by $g_t = \mathbf{F}_{t\delta}$, $t \in [0, 1]$. Notice that $t\delta|_C = 0$ as well, whence $\{g_t\}$ is also fixed on C . Therefore $h \in \mathcal{S}_C^{\text{id}}$.

Conversely, let $h \in \mathcal{S}_C^{\text{id}}$, so h is isotopic to id_{T^2} in \mathcal{S}^{id} via an isotopy $\{h_t\}_{t \in [0, 1]}$ fixed on C and such $h_0 = \text{id}_{T^2}$ and $h_1 = h$. Then $h_t = \mathbf{F}_{\delta_t}$, $t \in [0, 1]$, for some smooth function $\delta_t : T^2 \rightarrow \mathbb{R}$. Since C is a non-fixed trajectory of \mathbf{F} , it follows from [8, Theorem 25], that

the values of δ_t on C continuously depend on t . But each δ_t takes a constant integer value on C , and $\text{id}_{T^2} = \mathbf{F}_0$, whence

$$\eta(h) = \delta_1|_C = \delta_t|_C = \delta_0|_C = 0,$$

that is $h \in \ker \eta$. □

6.5. Inverse of boundary isomorphism $\partial_C^{-1} : \pi_0 \mathcal{S}_C \longrightarrow \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)$. Thus we have that both $\ker i_0$ and \mathcal{M} are isomorphic to \mathbb{Z} . By Lemma 6.4 $\ker i_0$ is generated by the homotopy class of the diffeomorphism

$$g = \mathbf{F}_\sigma = \mathbf{M}_\sigma \in \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(T^2, C) \subset \mathcal{S}_C$$

defined by Eq. (6.2) and satisfying $\eta(g) = -1$.

On the other hand, $q(\mathcal{M})$ is generated by the homotopy class of the following map:

$$q(\mathbf{M}) : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2}), \quad q(\mathbf{M})(t) = \mathbf{M}_t \quad (6.4)$$

Therefore in order to complete Proposition 4.3 it suffices to establish the following lemma.

Lemma 6.6. $i_1 \circ \partial_C^{-1}[g] = [q(\mathbf{M})]$. Hence $i_1 \circ \partial_C$ isomorphically maps $\ker i_0$ onto $q(\mathcal{M})$.

Proof. Recall that the boundary homomorphism $\partial_C : \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C) \longrightarrow \pi_0 \mathcal{S}_C$ is defined as follows: if $\alpha \in \pi_1(\mathcal{D}_C^{\text{id}}, \mathcal{S}_C)$ and $\omega : (I, \partial I, 0) \rightarrow (\mathcal{D}_C^{\text{id}}, \mathcal{S}_C, \text{id}_{T^2})$ is a representative of α , then

$$\partial_C(\alpha) = [\omega(1)] \in \pi_0 \mathcal{S}_C.$$

Now let $\mathbf{G}_t = \mathbf{M}_{t(\sigma+1)}$ be an isotopy between $\mathbf{G}(0) = \text{id}_{T^2}$ and $\mathbf{G}(1) = g$ fixed on C , see Eq. (6.3). Regard it as a map of triples $\mathbf{G} : (I, \partial I, 0) \longrightarrow (\mathcal{D}_C^{\text{id}}, \mathcal{S}_C, \text{id}_{T^2})$.

Then $\partial([\mathbf{G}]) = [\mathbf{G}(1)] = [g]$, and so

$$\partial_C^{-1}[g] = [\mathbf{G}].$$

As ∂_C is an isomorphism, $\partial_C^{-1}[g]$ does not depend on a particular choice of such an isotopy \mathbf{G} . Furthermore, $i_1 \circ \partial_C^{-1}[g]$ is a homotopy class of \mathbf{G} regarded as a map

$$\mathbf{G} : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2}), \quad \mathbf{G}(t) = \mathbf{M}_{t(\sigma+1)}. \quad (6.5)$$

Therefore it remains to show that $[\mathbf{G}] = [q(\mathbf{M})]$, that is the maps (6.4) and (6.5) are homotopic as maps of triples.

In fact the homotopy between them can be defined as follows:

$$\mathbf{H} : (I, \partial I, 0) \times I \longrightarrow (\mathcal{D}^{\text{id}}, \mathcal{S}, \text{id}_{T^2}), \quad \mathbf{H}(t, s) = \mathbf{M}_{t(s\sigma+1)}.$$

1) First we verify that $\mathbf{H}(t, s)$ is a diffeomorphism for all $t, s \in I$. As $g = \mathbf{M}_\sigma$ is a diffeomorphism, it follows from Lemma 3.8 that $\mathbf{M}_{s\sigma} = \mathbf{M}_{s\sigma+1}$ is also a diffeomorphism for all $s \in I$. But then by the same lemma $\mathbf{H}(t, s) = \mathbf{M}_{t(s\sigma+1)}$ is a diffeomorphism for all $t, s \in I$ as well.

2) Now let us show that \mathbf{H} is a homotopy of maps of triples. Indeed, for each $s \in I$ we have that $\mathbf{H}_{0,s} = \mathbf{M}_0 = \text{id}_{T^2}$, and $\mathbf{H}_{1,s} = \mathbf{M}_{s\sigma+1} = \mathbf{M}_{s\sigma} \in \mathcal{S}$.

3) Finally, $\mathbf{H}_{t,0} = \mathbf{M}_t = q(\mathbf{M})(t)$, and $\mathbf{H}_{t,1} = \mathbf{M}_{t(\sigma+1)} = \mathbf{G}(t)$ for all $t \in I$. Thus \mathbf{H} is a homotopy between $q(\mathbf{M})$ and \mathbf{G} . Lemma 6.6 and Proposition 4.3 are completed. □

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